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Extremes of weighted Dirichlet arrays

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Abstract In this paper we study the asymptotic behaviour of sample maxima of weighted Dirichlet triangular arrays. Two cases are interesting for our analysis, a) the associated random radius of the triangular array has distribution function in the Gumbel, b) or in the Weibull max-domain of attraction. In this paper we derive the asymptotic conditions that turn such arrays in Hüsler–Reiss triangular arrays.

Keywords Hüsler–Reiss triangular array · Weighted Dirichlet random vectors · Max-domain of attractions · Tail asymptotics · Asymptotic independence · Max-stable distribution

AMS 2000 Subject Classifications 60G70 · 60F05

1 Introduction

The asymptotic behaviour of multivariate sample extremes is a central topic of extreme value theory. Several results for large classes of multivariate distributions are known. For instance, it is well-known (see Sibuya 1960; Hüsler 1989a; Reiss 1989; Hüsler and Reiss 1989; Falk et al. 2004; Reiss and Thomas 2007; or Resnick 2008) that the multivariate Gaussian distribution is in the max-domain of attraction of a product distribution with Gumbel marginal distributions. The fact that the limiting distribution of the normalised sample maxima is a

Dedicated to Professor Jürg Hüsler on the occasion of his 60th birthday.

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product distribution means that the components of the sample maxima are asymptotically independent.

Despite that asymptotic independence is a nice property, in view of Sibuya's result we cannot use the multivariate Gaussian distribution for statistical modelling of asymptotically dependent sample maxima. To overcome that Hüsler and Reiss (1989) introduced a triangular array model which proved that the Gaussian setup is nonetheless useful to model asymptotically dependent sample extremes.

In order to explain the main idea of the aforementioned paper consider $\boldsymbol{S} := (S_1, S_2)^\top$ a bivariate standard Gaussian random vector, and let $\boldsymbol{X}_n^{(j)}, 1 \le j \le n, n \ge 1$, be an array of independent bivariate random vectors with stochastic representation

$$\boldsymbol{X}_n^{(j)} \stackrel{d}{=} (a_n S_1 + b_n S_2, S_2)^\top, \quad 1 \le j \le n, n \ge 1, \quad a_n, b_n \in \mathbb{R}.$$

Here $^\top$ stands for the transpose sign, and $\stackrel{d}{=}$ means equality of distribution functions.

Clearly, if $a_n \to 0, b_n \to 1$ as $n \to \infty$ the one dimensional projection $a_n S_1 + b_n S_2$ of $\boldsymbol{S}$ tends to S_2 almost surely, implying that $\boldsymbol{X}_n^{(j)} \to \boldsymbol{S}$ in probability $(n \to \infty)$ for any $j \in \mathbb{N}$. The construction of this triangular array via the two projections $a_n S_1 + b_n S_2$ and S_2 may eventually force the componentwise maxima to have asymptotic dependent components. As shown in Hüsler and Reiss (1989) a certain rate of converge to 0 for the deterministic sequence $a_n, n \ge 1$ is needed in order to imply asymptotic dependence. In the Gaussian setup $\mathrm{corr}(a_n S_1 + b_n S_2, S_2) = b_n, n \ge 1$. In view of Sibuya's result the convergence of correlations to 1 is a crucial restriction.

The Hüsler–Reiss Gaussian model can be naturally extended by taking $\boldsymbol{S}$ to be a spherical random vector. The triangular array of the bivariate random vectors $\boldsymbol{X}_n^{(j)}, 1 \le j \le n, n \ge 1$ consists thus of elliptical random vectors (see e.g., Cambanis et al. 1981; Fang et al. 1990), which by definition are linear transformations of spherical ones.

In the Hüsler–Reiss Gaussian model the associated random radius $R = \sqrt{S_1^2 + S_2^2}$ is chi-square distributed with 2 degrees of freedom; a basic asymptotic property related to the distribution function F of R is that it is in the Gumbel max-domain of attraction. Hashorva (2005) shows that the latter property ensures that the idea of Hüsler and Reiss is useful also for the elliptical setup. Surprisingly, even for elliptical triangular arrays the limiting distribution of the normalised sample maxima remains the same as in the Gaussian model, namely the Hüsler–Reiss distribution function (see Hashorva 2006c).

A natural generalisation of spherical random vectors is the class of Dirichlet random vectors (see Kotz et al. 2000). In Hashorva (2006d) it is shown that the limiting distribution of the sample maxima of bivariate Dirichlet triangular arrays is not always the Hüsler–Reiss distribution. Similar asymptotic results for the sample maxima can be stated if F is in the Weibull max-domain of attraction (see Hashorva 2006b, 2007b).

When the associated random radius R of a spherical random vector has distribution function in the Fréchet max-domain of attraction it is known (Hashorva et al. 2007) that the components of the random vector are asymptotically dependent, meaning that the sample maxima has asymptotically dependent components. Recent statistical applications of the Fréchet case are suggested in Klüppelberg et al. (2007).

Without going to mathematical details we describe briefly the main results of this contribution: Let $\boldsymbol{X}_n^{(j)}, 1 \le j \le n, n \ge 1$ be independent random vectors in $I\!R^k$, $k \ge 2$ defined by the stochastic representation

$$\boldsymbol{X}_n^{(j)} \stackrel{d}{=} A_n R \boldsymbol{W}, \quad 1 \le j \le n, \tag{1.1}$$

where $\boldsymbol{W} = (W_1, \ldots, W_k)^\top$ is a weighted Dirichlet random vector in $I\!R^k$ (see Definition 2.1 below) independent of the random variable $R > 0$ (almost surely), and $A_n, n \ge 1$ is a sequence of $k-$dimensional real square matrices. Let $\boldsymbol{a}_{ni}$ be the $i-$th row vector of A_n. Then $\boldsymbol{a}_{ni}\boldsymbol{W}$ is the $i-$th projection of $\boldsymbol{W}$ along the direction $\boldsymbol{a}_{ni}$. As in the Hüsler–Reiss model we consider the case of asymptotically singular projections of the random vector $\boldsymbol{W}$ assuming that almost surely

$$A_n \boldsymbol{W} \to (W_k, \ldots, W_k)^\top, \quad n \to \infty.$$

If the componentwise sample maxima $\boldsymbol{M}_n, n \ge 1$ of the triangular array in (1.1) converges in distribution (utilising a linear transformation) to a random vector with a non-degenerate max-infinitely divisible distribution (for short max-id), then we say that $\boldsymbol{X}_n^{(j)}, 1 \le j \le n, n \ge 1$ is a Hüsler–Reiss triangular array.

In this paper we show that if the associated random radius R has distribution function F in the Gumbel or the Weibull max-domain of attraction, then the above triangular array is a Hüsler–Reiss triangular array, provided that the sequence of the matrices $A_n, n \ge 1$ satisfies a certain asymptotic condition.

It is of some interest that for the Gumbel case (assumption on F) the limiting distribution of the sample maxima is max-stable, which is not the case when F is in the Weibull max-domain of attraction.

Organisation of the paper: In Section 2 we present some notation, introduce the family of weighted Dirichlet random vectors, and give some preliminary results. The case R has distribution function in the Gumbel max-domain of attraction is dealt with in Section 3. Similar results are shown in Section 4 assuming that R has distribution function in the Weibull max-domain of attraction. In Section 5 we give several illustrating examples. The proofs of all the results are relegated to Section 6.

2 Preliminaries

We shall introduce first some notation needed in the multivariate mathematical settings of the paper. Then we consider weighted Dirichlet distributions and related triangular arrays followed by few results from the extreme value theory.

Let I be a non-empty subset of $\{1, \ldots, k\}$, $k \geq 2$, and set $J := \{1, \ldots, k\} \setminus I$. For any vector $\boldsymbol{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k$ set $\boldsymbol{x}_I := (x_i, i \in I)^\top$. Further we shall define ($\boldsymbol{x}$, $\boldsymbol{y}$ are arbitrary vectors in $\mathbb{R}^k$)

$$
\begin{aligned}
&\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, \ldots, x_k + y_k)^\top, \\
&\boldsymbol{x} > \boldsymbol{y}, \text{ if } x_i > y_i, \quad \forall i = 1, \ldots, k, \\
&\boldsymbol{x} \geq \boldsymbol{y}, \text{ if } x_i \geq y_i, \quad \forall i = 1, \ldots, k, \\
&\boldsymbol{x} \neq \boldsymbol{y}, \text{ if } x_i \neq y_i \text{ for some } i \leq k, \\
&\boldsymbol{x} \not\leq \boldsymbol{y}, \text{ if } x_i > y_i \text{ for some } i \leq k,
\end{aligned}
$$

$$
\boldsymbol{a}\boldsymbol{x} := (a_1 x_1, \ldots, a_k x_k)^\top, \quad c\boldsymbol{x} := (cx_1, \ldots, cx_k)^\top, \quad \boldsymbol{a} \in \mathbb{R}^k, c \in \mathbb{R},
$$
$$
\boldsymbol{0} := (0, \ldots, 0)^\top \in \mathbb{R}^k, \quad \boldsymbol{1} := (1, \ldots, 1)^\top \in \mathbb{R}^k.
$$

For notational simplicity we write $\boldsymbol{x}_I^\top$ instead of $(\boldsymbol{x}_I)^\top$.

If the random vector $\boldsymbol{Y}$ has the distribution function H, we shall denote this by $\boldsymbol{Y} \sim H$. Further, $\mathcal{I} \sim Be(q)$ means that $\mathcal{I}$ is a random variable taking values in $\{-1, 1\}$ with $\boldsymbol{P}\{\mathcal{I} = 1\} = q \in (0, 1]$. In our notation the Beta distribution with positive parameters a, b possess the density function $x^{a-1}(1-x)^{b-1}\Gamma(a+b)/(\Gamma(a)\Gamma(b))$, $x \in (0, 1)$, with $\Gamma(\cdot)$ the Gamma function. We shall be denoting by $Beta(a, b)$ and $Gamma(a, b)$ their corresponding distribution functions.

In Hashorva et al. (2007) distributional and asymptotical properties of L_p-norm generalised symmetrised Dirichlet random vectors are discussed. In view of the amalgamation property shown in Theorem 2 therein a random vector $\boldsymbol{S}$ in $\mathbb{R}^k$ has L_p-norm generalised symmetrised Dirichlet distribution ($p > 0$) with positive parameters $\alpha_1, \ldots, \alpha_k$ and non-degenerate distribution function F such that $F(0) = 0$ iff it possesses the stochastic representation

$$
\boldsymbol{S} \stackrel{d}{=} R\boldsymbol{W}, \tag{2.1}
$$

with R, $\boldsymbol{W}$ independent, $R \sim F$ and $\boldsymbol{W}$ with stochastic representation

$$
\boldsymbol{W} \stackrel{d}{=} \left(\mathcal{I}_1(1 - V_1^p)^{1/p}, \ldots, \mathcal{I}_{k-1}\left(1 - V_{k-1}^p\right)^{1/p} \prod_{i=1}^{k-2} V_i, \mathcal{I}_k \prod_{i=1}^{k-1} V_i \right),
$$

where $\mathcal{I}_1, \ldots, \mathcal{I}_k, V_1, \ldots, V_{k-1}$ are independent random variables such that

$$
\mathcal{I}_i \sim Be(1/2), \; 1 \leq i \leq k, \; V_i > 0, V_i^p \sim Beta(\alpha_{i+1} + \cdots + \alpha_k, \alpha_i), \; 1 \leq i \leq k-1.
$$

L_2-norm generalised symmetrised Dirichlet random vectors are introduced in Fang and Fang (1990). If $p > 0$ and $\alpha_i = 1/p, 1 \leq i \leq k$, then $\boldsymbol{S} \stackrel{d}{=} R\boldsymbol{W}$ has a L_p-norm spherical distribution, see Gupta and Song (1997) or Szabłowski (1998). In this paper we focus on a larger class of random vectors which we refer to as the class of weighted Dirichlet random vectors. The spherical random vectors are contained in this class for special choice of parameters.

Definition 2.1 (Weighted Dirichlet Random Vectors) Let F be a distribution function with upper endpoint $x_F \in (0, \infty]$ satisfying $F(0) = 0$, and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ be given vectors in $(0, \infty)^{k-1}$. A random vector $\boldsymbol{X}$ in $\mathbb{R}^k, k \geq 2$, is said to possess a weighted Dirichlet distribution with parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, F$ if

$$\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{W}, \tag{2.2}$$

where $R \sim F$ is independent of the random vector $\boldsymbol{W}$ defined by

$$\boldsymbol{W} := \left(\mathcal{I}_1 \left(1 - V_1^{p_1}\right)^{1/r_1}, \ldots, \mathcal{I}_{k-1} \left(1 - V_{k-1}^{p_{k-1}}\right)^{1/r_{k-1}}) \prod_{i=1}^{k-2} V_i, \mathcal{I}_k \prod_{i=1}^{k-1} V_i\right), \tag{2.3}$$

with

$$\mathcal{I}_i \sim Be(q_i), \quad 1 \leq i \leq k, \quad V_i > 0,\ V_i^{p_i} \sim Beta(a_i, b_i), \quad 1 \leq i \leq k-1,$$

and $\mathcal{I}_1, \ldots, \mathcal{I}_k, V_1, \ldots, V_{k-1}$ are independent.

In the following we consider for simplicity only the case

$$q_i = 1, \quad 1 \leq i \leq k-1. \tag{2.4}$$

The general case can be treated with similar arguments (see Example 2 and Example 4 below). Under this restriction on the parameters we write for $\boldsymbol{X}$ defined above

$$\boldsymbol{X} \sim \mathcal{WD}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, F)$$

to mean that $\boldsymbol{X}$ is a weighted Dirichlet random vector ($\boldsymbol{q} = (1, \ldots, 1)^\top$).

Let $A_n, n \geq 1$ be a sequence of $k \times k$ real matrices and denote by $a_{n,ij}$ the ij–th entry of A_n. As in the Hüsler–Reiss model we consider a triangular array $\boldsymbol{X}_n^{(j)}, 1 \leq j \leq n, n \geq 1$ of random vectors such that

$$\boldsymbol{X}_n^{(j)} \stackrel{d}{=} A_n \boldsymbol{X}, \qquad 1 \leq j \leq n, n \geq 1, \quad \text{with } \boldsymbol{X} \sim \mathcal{WD}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, F). \tag{2.5}$$

Assuming further

$$\lim_{n\to\infty} a_{n,ij} = 0, \text{ and } \lim_{n\to\infty} a_{n,ik} = 1, \quad \forall 1 \leq i \leq k, 1 \leq j \leq k-1 \tag{2.6}$$

we obtain the almost sure convergence $A_n \boldsymbol{X} \to (X_k, \ldots, X_k)^\top, n \to \infty$.

Next, define the componentwise maxima of the above triangular array by

$$M_{n1} := \max_{1\leq j\leq n} X_{n1}^{(j)}, \ldots, M_{nk} := \max_{1\leq j\leq n} X_{nk}^{(j)}. \tag{2.7}$$

Our main interest in this paper is the asymptotic behaviour of the maxima $\boldsymbol{M}_n := (M_{n1}, \ldots, M_{nk})^\top, n \geq 1$ focusing on two situations i) the random radius R has distribution function F in the Gumbel max-domain of attraction, and ii) F is in the Weibull max-domain of attraction.

For more details the reader might consult the following extreme value theory monographs: De Haan (1970), Leadbetter et al. (1983), Galambos (1987), Reiss (1989), Falk et al. (2004), Kotz and Nadarajah (2005), de Haan and Ferreira (2006), or Resnick (2008).

The main motivation for our results stems from Hüsler–Reiss idea in the setup of Gaussian triangular arrays, which leads us to the following definition:

Definition 2.2 (Hüsler–Reiss Triangular Arrays) Let $\boldsymbol{X}_n^{(j)}, 1 \le j \le n, n \ge 1$ be a triangular array of $k-$dimensional random vectors with pertaining distribution function F_n, and let $\boldsymbol{M}_n$ denote the componentwise maxima defined by (2.7). If almost surely as $n \to \infty$

$$X_{ni}^{(1)} \to X, \quad \forall 1 \le i \le k$$

and further the convergence in distribution

$$\frac{\boldsymbol{M}_n - \boldsymbol{b}_n}{\boldsymbol{a}_n} \xrightarrow{d} \boldsymbol{\mathcal{M}}, \quad n \to \infty \tag{2.8}$$

holds with constants $\boldsymbol{a}_n > \boldsymbol{0}, \boldsymbol{b}_n \in I\!R^k$ and $\boldsymbol{\mathcal{M}} \sim H$, where H is a non-degenerated max-id distribution function, then we call the above array a Hüsler–Reiss triangular array with limiting distribution function H.

In our setup, since we assume (2.6), the triangular array of interest (defined in (2.5)) is a Hüsler–Reiss triangular array if additionally (2.8) holds. We show that under an asymptotic condition on A_n for both Gumbel and the Weibull case (2.8) can be stated, thus confirming that the triangular array of interest is a Hüsler–Reiss one.

3 Gumbel max-domain of attraction

In this section we investigate the asymptotic behaviour of the sample maxima considering a triangular array of weighted Dirichlet random vectors introduced previously, assuming further that the distribution function F of the associated random radius R is in the max-domain of attraction of the Gumbel distribution $\Lambda(x) := \exp(-\exp(-x)), x \in I\!R$. This means that (see e.g. Reiss 1989 or Falk et al. 2004)

$$\lim_{u \uparrow x_F} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in I\!R, \tag{3.1}$$

where $x_F := \sup\{s : F(s) < 1\}$ is the upper endpoint of the distribution function F and w is a scaling function defined asymptotically by

$$w(u) := \frac{(1 + o(1))[1 - F(u)]}{\int_u^{x_F} [1 - F(s)]\, ds}, \quad u \uparrow x_F. \tag{3.2}$$

If (3.1) holds, then we write in the sequel $F \in MDA(\Lambda, w)$.

Next, we deal with the tail asymptotic behaviour of a one dimensional projection of a weighted Dirichlet random vector $\boldsymbol{X} \sim \mathcal{WD}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, F)$ in $I\!R^k, k \ge 2$. The main result of this section is presented then in Theorem 3.4 below.

For $\boldsymbol{a}_n, n \geq 1$ given vectors in $\mathbb{R}^k$ we define the projection of $\boldsymbol{X}$ along $\boldsymbol{a}_n$ by

$$Z_n := \boldsymbol{a}_n^\top \boldsymbol{X} = \sum_{i=1}^{k} a_{ni} X_i, \quad n \geq 1.$$

Assuming

$$\lim_{n\to\infty} a_{n1} = \cdots = \lim_{n\to\infty} a_{n,k-1} = 0, \quad \lim_{n\to\infty} a_{nk} = 1 \tag{3.3}$$

we retrieve the almost sure convergence $Z_n \to X_k, n \to \infty$. Intuitively, the asymptotic tail behaviour ($n \to \infty$) of Z_n is strongly related to that of X_k. In view of Theorem 6.2 in Hashorva (2007c) the tail asymptotic behaviour of X_k is known if F is in the Gumbel max-domain of attraction. Under the latter assumption we show in the next theorem that the tail asymptotic behaviour of Z_n is the same as that of X_k (up to a constant).

Theorem 3.1 *Let $\boldsymbol{X} \sim \mathcal{WD}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, F)$ be a k-dimensional weighted Dirichlet random vector with positive associated random radius $R \sim F$. Let $y_n, n \geq 1$ be given constant such that*

$$\lim_{n\to\infty} y_n = x_F, \text{ and } |y_n| < x_F, \quad \forall n \geq 1. \tag{3.4}$$

Assume that $F \in MDA(\Lambda, w)$ with w some positive scaling function. If $\boldsymbol{a}_n$, $n \geq 1$ is a sequence of vectors in $\mathbb{R}^k$ satisfying (3.3) and further

$$\lim_{n\to\infty} (1 - a_{nk}) h_n = \lambda_k \in \mathbb{R}, \quad \lim_{n\to\infty} a_{nj} h_n^{1-1/r_i} = \lambda_j \in \mathbb{R}, \quad 1 \leq j \leq k-1 \tag{3.5}$$

holds with $h_n := y_n w(y_n), n \geq 1$, then we have as $n \to \infty$

$$\boldsymbol{P}\left\{\sum_{i=1}^{k} a_{ni} X_i > y_n\right\}$$

$$= (1 + o(1)) K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) \boldsymbol{P}\{X_k > y_n\} \tag{3.6}$$

$$= (1 + o(1)) K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) \prod_{i=1}^{k} \frac{p_i^{b_i} \Gamma(a_i + b_i)}{\Gamma(a_i)} (y_n w(y_n))^{-\sum_{i=1}^{k-1} b_i} [1 - F(y_n)], \tag{3.7}$$

where (set $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_k)^\top \in \mathbb{R}^k$)

$$K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) := \exp(-\lambda_k) \prod_{j=1}^{k-1} \left[\frac{1}{p_j^{b_j} \Gamma(b_j)} \times \int_0^\infty \exp\left(\lambda_j s^{1/r_j} - s/p_j\right) s^{b_j - 1}\, ds \right] \in (0, \infty). \tag{3.8}$$

Corollary 3.2 *Under the assumptions of Theorem* 3.1*, if furthermore* Z_{nj}*,* $1 \le j \le n, n \ge 1$ *is a triangular array of independent random variables satisfying*

$$Z_{nj} \stackrel{d}{=} \sum_{i=1}^{k} a_{ni} X_i, \quad 1 \le j \le n, n \ge 1,$$

then we have

$$\frac{\max_{1\le j\le n} Z_{nj} - d_n}{c_n} - \ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda})) \stackrel{d}{\to} Y \sim \Lambda, \quad n \to \infty, \tag{3.9}$$

with $d_n := G^{-1}(1 - 1/n), c_n := 1/w(d_n), n > 1,$ *and* G^{-1} *the inverse of the distribution function of* X_k*.*

The convergence above implies that $M_{nj}, 1 \le j \le k, n \ge 1$ converges in distribution to a Gumbel random variable, provided that F is in the Gumbel max-domain of attraction. The joint convergence in distribution for the random sequence $\boldsymbol{M}_n, n \ge 1$ is shown in Theorem 3.4 below.

Remark 3.3

a) If the univariate distribution function F satisfies (3.1), then we have

$$\lim_{u \uparrow x_F} k(u)w(u) = \infty, \tag{3.10}$$

with $k(u) := u$ if $x_F = \infty$ and $k(u) := x_F - u$ otherwise. Consequently, (3.4) implies

$$\lim_{n\to\infty} y_n w(y_n) = \infty, \tag{3.11}$$

hence if $\lambda_i \neq 0, r_i > 1$ hods for some $i \le k$, then by (3.3) necessarily $\lim_{n\to\infty} a_{ni} = 0$ and $\lim_{n\to\infty} a_{nk} = 1$ for $i = k$.

b) If $\boldsymbol{\lambda} = \boldsymbol{0} \in \mathbb{R}^k$, then we have $K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) = 1$, consequently (3.6) implies

$$\boldsymbol{P}\left\{\sum_{i=1}^{n} a_{ni} X_i > y_n\right\} = (1 + o(1))\boldsymbol{P}\{X_k > y_n\}, \quad n \to \infty,$$

which is obviously true when $a_{ni} = 0, 1 \le i \le k-1, a_{nk} = 1$. If for some $i < k$ we have $r_i \le 1$, then (3.3) yields $\lambda_i = 0$. Note in passing that $K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda})$ does not depend on the parameter $\boldsymbol{a}$.

If $\boldsymbol{X}$ is a random vector in $\mathbb{R}^k, k \ge 2$ we define a triangular array of random vectors $\boldsymbol{X}_n^{(j)} = (X_{n1}^{(j)}, \dots, X_{nk}^{(j)})^\top, 1 \le j \le n, n \ge 1$ as in (2.5) with $A_n := \{a_{n,ij}\}_{i,j}, \in \mathbb{R}^{k\times k}, n \ge 1$. The maxima of this triangular array has components

$$M_{ni} := \max_{1\le j\le n} X_{ni}^{(j)}, \quad i = 1, \dots, k.$$

Next, we formulate the result for the triangular array setup showing the necessary conditions for it to be a Hüsler–Reiss triangular array.

Theorem 3.4 *Let* $F, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{X}, w, c_n, d_n, n \geq 1$ *be as in Corollary* 3.2, *and let* $\boldsymbol{X}_n^{(j)} = (X_{n1}^{(j)}, \ldots, X_{nk}^{(j)})^\top, 1 \leq j \leq n, n \geq 1$ *be a triangular array defined via* (2.5) *with* $A_n := \{a_{n,ij}\}_{i,j}, \in I\!R^{k\times k}, n \geq 1$ *a sequence of square matrices satisfying* (2.6). *If* $F \in MDA(\Lambda, w)$ *and furthermore*

$$\lim_{n\to\infty}(1-a_{n,ik})h_n = \lambda_{ik} \in I\!R,$$

$$\lim_{n\to\infty} a_{n,ij}h_n^{1-1/r_j} = \lambda_{ij} \in I\!R, \quad 1 \leq i \leq k, 1 \leq j \leq k-1 \tag{3.12}$$

holds with $h_n := d_n/c_n, n > 1$, *then we have the convergence in distribution*

$$\left(\frac{M_{n1}-d_n}{c_n} - \ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_1)), \ldots, \frac{M_{nk}-d_n}{c_n} - \ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_k))\right)$$

$$\stackrel{d}{\to} \boldsymbol{\mathcal{M}} \sim H, \quad n\to\infty, \tag{3.13}$$

with $K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l), \boldsymbol{\lambda}_l := (\lambda_{l1}, \ldots, \lambda_{lk})^\top, 1 \leq l \leq k$, *defined in* (3.8) *and* H *defined by*

$$H(\boldsymbol{x}) := \exp\left(-\sum_{i=1}^{k}(-1)^{i+1}\sum_{|L|=i}\int_0^\infty\cdots\int_0^\infty \min_{l\in L} A_l(x_l, s_1, \ldots, s_{k-1}) \right.$$
$$\left.\times\prod_{j=1}^{k-1} s_j^{b_j-1}\, ds_1\cdots ds_{k-1}\right), \quad \boldsymbol{x} \in I\!R^k, \tag{3.14}$$

where the summation above is over all index subsets L *of* $\{1, \ldots, d\}$ *and the function* $A_l, l \in L$ *is defined by*

$$A_l(y, s_1, \ldots, s_{k-1})$$
$$:= \prod_{j=1}^{k-1}\left[\frac{1}{p_j^{b_j}\Gamma(b_j)}\right]\exp\left(\sum_{j=1}^{k-1}\left[\lambda_{lj}s_j^{1/r_j} - s_j/p_j\right] - y\right.$$
$$\left. - \lambda_{lk} - \ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l))\right), \quad y \in I\!R. \tag{3.15}$$

The limiting distribution function H specified in (3.14) is max-stable with unit Gumbel marginal distributions, which follows easily since for any $y \in I\!R$ and $l \in \{1, \ldots, k\}$

$$A_l(y+\ln t, s_1, \ldots, s_{k-1}) = A_l(y, s_1, \ldots, s_{k-1})/t, \quad \forall s_i > 0, i \leq k-1, \quad \forall t > 0$$

implying

$$(H(x_1+\ln t, \ldots, x_k+\ln t))^t = H(x_1, \ldots, x_k), \quad \forall (x_1, \ldots, x_k) \in I\!R^k, \quad \forall t > 0.$$

See Galambos (1987), Hüsler (1989b), Falk et al. (2004), Reiss and Thomas (2007), or Resnick (2008) for the main properties of the max-stable distribution functions. It is interesting that in the triangular array setup of Hüsler and Reiss the max-stability property is preserved. If the associated random radius R has distribution function in the Weibull max-domain of attraction this is no longer the case, as will be shown in the next section.

4 Weibull max-domain of attraction

The main assumption in this section is that the associated random radius R has distribution function F in the Weibull max-domain of attraction, i.e., for some $\gamma \in (0, \infty)$ we have

$$\lim_{u \downarrow 0} \frac{1 - F(1 - ux)}{1 - F(1 - u)} = x^{\gamma}, \quad \forall x \in (0, \infty). \tag{4.1}$$

The upper endpoint x_F of F is necessarily finite. We assume for simplicity in the following that $x_F = 1$. Write next x_+ instead of $\max(0, x)$, $x \in \mathbb{R}$, and denote by $\mathbf{\Psi}_{\gamma}(x) = \exp(-|x|^{\gamma})$, $x < 0$ the unit Weibull distribution with index $\gamma \in (0, \infty)$. As in the Gumbel case, we deal first with the tail asymptotics of a simple projection of a weighted Dirichlet random vector. Then we present the main result of this section in Theorem 4.3 below.

Theorem 4.1 *Let $F, \boldsymbol{\lambda}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{X}, a_{ni}, n \geq 1, 1 \leq i \leq k-1$ be as in Theorem* 3.1. *Assume that the distribution function F with $x_F = 1$ is in the max-domain of attraction of $\Psi_{\gamma}, \gamma \in (0, \infty)$. If $h_n, n \geq 1$ are constants satisfying $\lim_{n\to\infty} h_n = \infty$ and* (3.5) *holds, then as $n \to \infty$*

$$\left\{ \sum_{i=1}^{k} a_{ni} X_i > 1 - 1/h_n \right\}$$
$$= (1 + o(1)) C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) \boldsymbol{P} \{X_k > 1 - 1/h_n\} \tag{4.2}$$
$$= (1 + o(1)) C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) \left[\prod_{i=1}^{k-1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)\Gamma(b_i)} \right] h_n^{-\sum_{i=1}^{k-1} b_i} [1 - F(1 - 1/h_n)], \tag{4.3}$$

where (set $\gamma_i^* := 1 + \gamma + \sum_{j=i}^{k-1} b_j$)

$$C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) := \left[\prod_{i=1}^{k-1} \frac{\Gamma(\gamma_i^*)}{p_i^{b_i} \Gamma(\gamma_i^* - b_i) \Gamma(b_i)} \right]$$
$$\times \int_0^{\infty} \cdots \int_0^{\infty} \left(1 - \lambda_k - \sum_{i=1}^{k-1} s_i / p_i + \sum_{i=1}^{k-1} \lambda_i s_i^{1/r_i} \right)_+^{\gamma}$$
$$\times \prod_{i=1}^{k-1} s_i^{b_i - 1} \, ds_1 \cdots ds_{k-1} \in [0, \infty). \tag{4.4}$$

Corollary 4.2 *Under the assumptions of Theorem* 3.1, *if furthermore $Z_{nj}, 1 \leq j \leq n, n \geq 1$ is a triangular array of independent random variables satisfying*

$Z_{nj} \stackrel{d}{=} \sum_{i=1}^{k} a_{ni} X_i, 1 \le j \le n, n \ge 1$, and $c_n := 1 - G^{-1}(1-1/n), n > 1$ with G^{-1} the inverse of the distribution function of X_k, then we have

$$\frac{\max_{1\le j\le n} Z_{nj} - 1}{c_n} \stackrel{d}{\to} \mathcal{M}, \quad n \to \infty, \tag{4.5}$$

where the random variable $\mathcal{M}$ has distribution function (set $\boldsymbol{x}^* := (|x|^{1/r_1 - 1}, \ldots, |x|^{1/r_{k-1}-1}, |x|^{-1})$)

$$Q_{\gamma,\boldsymbol{b},\boldsymbol{p},\boldsymbol{r},\boldsymbol{\lambda}}(x) = \exp\Big(-C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}\boldsymbol{x}^*)|x|^{\gamma + \sum_{i=1}^{k-1} b_i}\Big), \quad \forall x \in (-\infty, 0). \tag{4.6}$$

We state next the main result of this section.

Theorem 4.3 *Let $F, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{X}, A_n, n \ge 1$ and $\boldsymbol{X}_n^{(j)}, 1 \le j \le n, n \ge 1$ be as in Theorem 3.4. Assume that F is in the max-domain of attraction of $\Psi_\gamma, \gamma \in (0, \infty)$ with $x_F = 1$. Let further $c_n, n \ge 1$ be as in Corollary 4.2. If (3.12) holds, then we have the convergence in distribution*

$$\left(\frac{M_{n1} - 1}{c_n}, \ldots, \frac{M_{nk} - 1}{c_n}\right) \stackrel{d}{\to} \boldsymbol{\mathcal{M}} \sim H, \quad n \to \infty, \tag{4.7}$$

with H defined in (3.14) and (set $\gamma_i^* := 1 + \gamma + \sum_{j=i}^{k-1} b_j$)

$$A_l(x_l, s_1, \ldots, s_{k-1}) := \left[\prod_{i=1}^{k-1} \frac{\Gamma(\gamma_i^*)}{p_i^{b_i}\Gamma(\gamma_i^* - b_i)\Gamma(b_i)}\right] \left(|x_l| - \lambda_{lk} - \sum_{i=1}^{k-1} s_i/p_i + \sum_{i=1}^{k-1} \lambda_{li} s_i^{1/r_i}\right)_+^{\gamma}, \quad x_l \in (-\infty, 0].$$

Furthermore, $\mathcal{M}_l \sim Q_{\gamma,\boldsymbol{b},\boldsymbol{p},\boldsymbol{r},\boldsymbol{\lambda}_l}, 1 \le l \le k$ with $Q_{\gamma,\boldsymbol{b},\boldsymbol{p},\boldsymbol{r},\boldsymbol{\lambda}_l}$ as in (4.6) and $\boldsymbol{\lambda}_l := (\lambda_{l1}, \ldots, \lambda_{lk})^\top$.

Remark 4.4

a) If for all $i \le k-1$ we have $\lambda_i = 0$, then for any $\lambda_k < 1$ Lemma 15 in Hashorva et al. (2007) implies

$$C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) = \frac{1}{\Gamma(\sum_{i=1}^{k-1} b_i)} \left[\prod_{i=1}^{k-1} \frac{\Gamma(\gamma_i^*)}{\Gamma(\gamma_i^* - b_i)}\right] \times \int_0^\infty (1 - \lambda_k - x)_+^{\gamma} x^{\sum_{i=1}^{k-1} b_i - 1}\, dx \in (0, \infty). \tag{4.8}$$

b) The distribution function H in Theorem 4.3 is max-id but not max-stable.
c) In Hashorva (2007b) triangular arrays of bivariate Dirichlet random vectors with associated random radius in the Weibull max-domain of attraction are considered. Such arrays are special case of our setup when restricting $\boldsymbol{p} = (2, \ldots, 2)^\top \in I\!R^{k-1}, k \ge 2$. The multivariate elliptical case is dealt with in Hashorva (2008).

5 Examples

Next we present seven illustrating examples.

Example 1 Let $\boldsymbol{X} \sim \mathcal{WD}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, F)$ be a bivariate random vector as in Theorem 3.1 with constants

$$a_1 = a \in (0, \infty), \quad b_1 = b \in (0, \infty), \quad r_1 = p_1 = 2. \tag{5.1}$$

If (3.5) holds with $\lambda_1 = \sqrt{2\lambda}, \lambda_2 = \lambda \in [0, \infty)$, then we have (recall (3.8))

$$\begin{aligned} K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, (\sqrt{2\lambda}, \lambda)) &= \exp(-\lambda) \frac{1}{2^b \Gamma(b)} \int_0^\infty \exp\left(\sqrt{2\lambda s} - s/2\right) s^{b-1}\, ds \\ &= \frac{2}{2^b \Gamma(b)} \int_0^\infty \exp\left(-(t - \sqrt{2\lambda})^2/2\right) |t|^{2b-1}\, dt. \end{aligned}$$

If $\lambda_1 = -\sqrt{2\lambda}, \lambda_2 = \lambda \in [0, \infty)$, then we have

$$\begin{aligned} K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, (-\sqrt{2\lambda}, \lambda)) &= \exp(-\lambda) \frac{1}{2^b \Gamma(b)} \int_0^\infty \exp\left(-\sqrt{2\lambda s} - s/2\right) s^{b-1}\, ds \\ &= \frac{2}{2^b \Gamma(b)} \int_{-\infty}^0 \exp\left(-(t - \sqrt{2\lambda})^2/2\right) |t|^{2b-1}\, dt. \end{aligned}$$

Note that $b = 1/2$ implies $K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, (\sqrt{2\lambda}, \lambda)) + K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, (-\sqrt{2\lambda}, \lambda)) = 2$, thus not depending on λ. This is actually expected if (X_1, X_2) is spherically distributed where

$$p_1 = r_1 = 2, \quad a_1 = b_1 = q_1 = q_2 = 1/2 \tag{5.2}$$

since for any two constants d_1, d_2 Lemma 6.2 of Berman (1982) and (5.2) imply

$$d_1 X_1 + d_2 X_2 \stackrel{d}{=} X_1 \sqrt{d_1^2 + d_2^2}. \tag{5.3}$$

Example 2 Let $\boldsymbol{X} \sim \mathcal{WD}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, F)$ be a bivariate weighted Dirichlet random vector with parameters as in (5.1), and let $\mathcal{I}_i \sim Be(q_i), q_i \in (0, 1], i = 1, 2$ be two independent random variables being further independent of $\boldsymbol{X}$. The random vector $\boldsymbol{Y}$ defined below via the stochastic representation

$$\boldsymbol{Y} \stackrel{d}{=} (\mathcal{I}_1 X_1, \mathcal{I}_2 X_2)$$

is a bivariate weighted Dirichlet random vector with weights $\boldsymbol{q} := (q_1, q_2)^\top$. Given the constants $a_{n1}, a_{n2}, n \geq 1$ and $y_n \in (-x_F, x_F), n \geq 1$ with x_F the upper endpoint of F we have for $n > 1$

$$\begin{aligned} &\boldsymbol{P}\{a_{n1} Y_1 + a_{n2} Y_2 > y_n\} \\ &\quad = \sum_{1 \leq i \leq 2, 1 \leq j \leq 2} \boldsymbol{P}\{\mathcal{I}_1 = \varepsilon_i\} \boldsymbol{P}\left\{\mathcal{I}_2 = \varepsilon_j\right\} \boldsymbol{P}\left\{a_{n1} \varepsilon_i X_1 + a_{n2} \varepsilon_j X_2 > y_n\right\}, \end{aligned}$$

where $\varepsilon_i = \pm 1, i = 1, 2$. Assume that

$$\lim_{n \to \infty} a_{n1} = \lim_{n \to \infty} (1 - a_{n2}) = 0, \qquad \text{and } \lim_{n \to \infty} y_n = x_F.$$

If $x_F \in (0, \infty)$, then it follows easily that

$$\begin{aligned}
&\frac{\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > y_n\}}{\boldsymbol{P}\{X_2 > y_n\}}\\
&\quad = \boldsymbol{P}\{\mathcal{I}_1 = 1\}\,\boldsymbol{P}\{\mathcal{I}_2 = 1\}\,\frac{\boldsymbol{P}\{a_{n1}X_1 + a_{n2}X_2 > y_n\}}{\boldsymbol{P}\{X_2 > y_n\}}\\
&\qquad + \boldsymbol{P}\{\mathcal{I}_1 = -1\}\,\boldsymbol{P}\{\mathcal{I}_2 = 1\}\,\frac{\boldsymbol{P}\{-a_{n1}X_1 + a_{n2}X_2 > y_n\}}{\boldsymbol{P}\{X_2 > y_n\}} + o(1),\ n \to \infty.
\end{aligned}$$

The proof when $x_F = \infty$ and F is in the Gumbel max-domain of attraction is not trivial; it can be shown following the lines of the proof of Theorem 3.1. If $\boldsymbol{X}$, $a_{n1}, a_{n2}, y_n, n \ge 1$ satisfy the assumptions of Theorem 3.1 with $\lambda_1 = \sqrt{2\lambda}$, $\lambda_2 = \lambda \ge 0$ and $q_1 = 1/2$, then we obtain as $n \to \infty$

$$\begin{aligned}
&\frac{\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > y_n\}}{\boldsymbol{P}\{X_2 > y_n\}}\\
&\quad = q_1 q_2 2 \frac{1}{2^b\Gamma(b)} \int_0^\infty \exp\left(-\left(t - \sqrt{2\lambda}\right)^2/2\right)|t|^{2b-1}\,dt\\
&\qquad + (1 - q_1) q_2 2 \frac{1}{2^b\Gamma(b)} \int_{-\infty}^0 \exp\left(-\left(t - \sqrt{2\lambda}\right)^2/2\right)|t|^{2b-1}\,dt + o(1)\\
&\quad = q_2 \frac{1}{2^b\Gamma(b)} \int_{-\infty}^\infty \exp\left(-\left(t - \sqrt{2\lambda}\right)^2/2\right)|t|^{2b-1}\,dt + o(1).
\end{aligned}$$

Since for $p_1 = r_1 = 2$ the random variable X_2 has stochastic representation

$$X_2 \stackrel{d}{=} \left(1 - U_{a,b}^2\right)^{1/2}, \quad U_{a,b}^2 \sim Beta(b, a), \quad \boldsymbol{P}\left\{U_{a,b} \in [0, 1]\right\} = 1 \tag{5.4}$$

applying Theorem 12.3.1 of Berman (1992) we have (set $h_n := y_n w(y_n), n \ge 1$)

$$\boldsymbol{P}\{X_2 > y_n\} = (1 + o(1))2^b \frac{\Gamma(a + b)}{\Gamma(a)} h_n^{-b}[1 - F(y_n)], \quad n \to \infty.$$

Consequently

$$\begin{aligned}
\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > y_n\} &= (1 + o(1))h_n^{-b}[1 - F(y_n)]q_2 \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}\\
&\quad \times \int_{-\infty}^\infty \exp\left(-\left(t - \sqrt{2\lambda}\right)^2/2\right)|t|^{2b-1}\,dt,
\end{aligned}$$

which is proved in Theorem 2.2 in Hashorva (2006d) for $q_2 = 1/2$. If further $b = 1/2$

$$\begin{aligned}
&\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > y_n\}\\
&\quad = (1 + o(1))h_n^{-1/2} q_2 \sqrt{2} \frac{\Gamma(a_1 + 1/2)}{\Gamma(a_1)}[1 - F(y_n)], n \to \infty.
\end{aligned} \tag{5.5}$$

Note in passing that the asymptotics in (5.5) does not depend on λ.

Example 3 In this example we obtain a simpler formula for the limiting distribution function H derived in Theorem 3.4 provided that the constants are chosen as follows:

$$\lambda_{lk} := \sum_{j=1}^{k-1} \gamma_{lj}, \quad 1 \le l \le k,$$

$$\lambda_{ij} := \sqrt{2\gamma_{ij}}, \gamma_{ij} > 0, \quad p_i = r_i = 2, \quad i, j = 1, \ldots, k-1, k \ge 2.$$

For any subset L of $\{1, \ldots, k\}$ with at least two elements and any $\boldsymbol{x} \in I\!R^k$ (set $\bar{K}_l := \ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l))$)

$$\int_0^\infty \cdots \int_0^\infty \min_{l \in L} \left[\exp\left(\left[\sum_{j=1}^{k-1} \left[\lambda_{lj} s_j^{1/r_j} - s_j/p_j \right] - x_l - \lambda_{lk} - \bar{K}_l \right) \right] \right.$$
$$\times \prod_{j=1}^{k-1} s_j^{b_j - 1} \, ds_1 \cdots ds_{k-1}$$
$$= \int_0^\infty \cdots \int_0^\infty \min_{l \in L} \left[\exp\left(\sum_{j=1}^{k-1} \left[\sqrt{2\gamma_{lj} s_j} - s_j/2 - \gamma_{lj} \right] - x_l - \bar{K}_l \right) \right]$$
$$\times \prod_{j=1}^{k-1} s_j^{b_j - 1} \, ds_1 \cdots ds_{k-1}$$
$$= 2^{k-1} \int_0^\infty \cdots \int_0^\infty \min_{l \in L} \left[\exp\left(- \sum_{j=1}^{k-1} \left[s_j - \sqrt{2\gamma_{lj}} \right]^2 /2 - x_l - \bar{K}_l \right) \right]$$
$$\times \prod_{j=1}^{k-1} s_j^{2b_j - 1} \, ds_1 \cdots ds_{k-1}.$$

Plugging in (3.14) we obtain a simpler formula for H. If $\gamma_{lj} = \gamma \ge 0, 1 \le j \le k-1, 1 \le l \le k$, then the random vector $\boldsymbol{\mathcal{M}} \sim H$ in Theorem 3.4 has completely dependent unit Gumbel components.

Example 4 Let $F, \boldsymbol{X}, \boldsymbol{Y}$ be as in Example 2, and let $a_{n,11}, a_{n,12}, a_{n,21}, a_{n,22}$, $n \ge 1$ be four sequences of constants. Define a bivariate triangular array $\left(V_{n1}^{(j)}, V_{n2}^{(j)}\right)$, $1 \le j \le n, n \ge 1$ via the stochastic representation

$$V_{ni}^{(j)} \stackrel{d}{=} a_{n,i1} Y_1 + a_{n,i2} Y_2, \quad i = 1, 2, \ 1 \le j \le n, n \ge 1.$$

Assume that $F \in MDA(\Lambda, w)$ and the condition (3.12) holds with

$$\lambda_{11} = \lambda_{12} = 0, \ \lambda_{21} = \sqrt{2\gamma}, \ \lambda_{22} = \gamma \in (0, \infty), \ p_1 = r_1 = 2, \ q_1 = q_2 = b = 1/2.$$

For any sequence $y_n, n \geq 1$ such that $y_n < x_F, n \geq 1$ and $\lim_{n\to\infty} y_n = x_F$ we obtain (recall (5.5))

$$\boldsymbol{P}\left\{V_{ni}^{(1)} > y_n\right\} = (1 + o(1))[1 - G(y_n)], \quad n \to \infty, \quad i = 1, 2,$$

with G the distribution function of Y_1. If $d_n := G^{-1}(1 - 1/n), c_n := 1/w(d_n)$, $n > 1$ we may write for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\boldsymbol{P}\left\{V_{ni}^{(1)} > c_n x + d_n\right\}}{\boldsymbol{P}\left\{V_{ni}^{(1)} > d_n\right\}} = \exp(-x), \quad i = 1, 2.$$

If $M_{ni} := \max_{1\leq j\leq n} V_{ni}^{(j)}, n \geq 1, i = 1, 2$ is the componentwise maxima, then we have

$$\left(\frac{M_{n1} - d_n}{c_n}, \frac{M_{n2} - d_n}{c_n}\right) \xrightarrow{d} (\mathcal{M}_1, \mathcal{M}_2) \sim H, \quad n \to \infty,$$

where

$$H(x, y) = \exp(-\exp(-x) - \exp(-y) - A(x, y)), \quad x, y \in \mathbb{R},$$

with

$$\begin{aligned} A(x, y) = \frac{1}{2\sqrt{2}\Gamma(1/2)} \int_0^\infty \min\Big[&\exp(-s/2 - x), \\ &\exp\left(\sqrt{2\gamma s} - s/2 - y - \gamma\right)\Big] s^{-1/2}\, ds \\ + \frac{1}{2\sqrt{2}\Gamma(1/2)} \int_0^\infty \min\Big[&\exp(-s/2 - x), \\ &\exp\left(-\sqrt{2\gamma s} - s/2 - y - \gamma\right)\Big] s^{-1/2}\, ds, \; x, y \in \mathbb{R}. \end{aligned}$$

We show that the Hüsler–Reiss formula holds for H. For any $x, y \in \mathbb{R}$ we have

$$\begin{aligned} A(x, y) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \min\left[\exp\left(-s^2/2 - x\right), \exp\left(\sqrt{2\gamma} s - s^2/2 - y - \gamma\right)\right] ds \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty \min\left[\exp\left(-s^2/2 - x\right), \exp\left(-\sqrt{2\gamma} s - s^2/2 - y - \gamma\right)\right] ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \min\left[\exp\left(-s^2/2 - x\right), \exp\left(-\left(s - \sqrt{2\gamma}\right)^2/2\right)\right] ds \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty \min\left[\exp\left(-s^2/2 - x\right), \exp\left(-\left(s + \sqrt{2\gamma}\right)^2/2\right)\right] ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \min\left[\exp\left(-s^2/2 - x\right), \exp\left(-\left(s - \sqrt{2\gamma}\right)^2/2\right)\right] ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{\exp(-x)}{\sqrt{2\pi}} \int_{(y-x)/\sqrt{2\gamma}+\sqrt{\gamma/2}}^{\infty} \exp\left(-s^2/2\right)\, ds \\
&\quad + \frac{\exp(-y)}{\sqrt{2\pi}} \int_{-\infty}^{(y-x)/\sqrt{2\gamma}+\sqrt{\gamma/2}} \exp\left(-\left(s-\sqrt{2\gamma}\right)^2/2\right)\, ds \\
&= \exp(-x)\left[1-\Phi\left((y-x)/\sqrt{2\gamma}+\sqrt{\gamma/2}\right)\right] \\
&\quad + \frac{\exp(-y)}{\sqrt{2\pi}} \int_{(x-y)/\sqrt{2\gamma}-\sqrt{\gamma/2}+\sqrt{2\gamma}}^{\infty} \exp(-s^2/2)\, ds \\
&= \exp(-x)\left[1-\Phi\left((y-x)/\sqrt{2\gamma}+\gamma/2\right)\right] \\
&\quad + \exp(-y)\left[1-\Phi\left((x-y)/\sqrt{2\gamma}+\sqrt{\gamma/2}\right)\right],
\end{aligned}
$$

where Φ is the standard Gaussian distribution function on $I\!R$. Hence we have

$$
\begin{aligned}
-\ln(H(x,y)) = \exp(-x)\Phi\left(\frac{y-x}{\sqrt{2\gamma}}+\sqrt{\gamma/2}\right) \\
+ \exp(-y)\Phi\left(\frac{x-y}{\sqrt{2\gamma}}+\sqrt{\gamma/2}\right), \; x, y \in I\!R,
\end{aligned}
$$

which is derived in Hüsler and Reiss (1989) (see also Hüsler 1994; Falk and Reiss 2004; Hashorva 2005; Reiss and Thomas 2007). The distribution function H appears in the context of the extremes of Brownian motion in Brown and Resnick (1977), and in Gale (1980), Eddy and Gale (1981) in the context of extremes of convex hulls.

We present next three examples related to Section 4.

Example 5 Let us consider the calculation of the constant $C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda})$ appearing in (4.4) in the bivariate setup where the parameters are as in (5.1). If $\lambda_1 = \sqrt{2\lambda}$ and $\lambda_2 = \lambda \geq 0$, then we have

$$
\begin{aligned}
C\left(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, (\sqrt{2\lambda}, \lambda)\right) &= \frac{\Gamma(b+1+\gamma)}{2^b\,\Gamma(\gamma+1)\Gamma(b)} \int_0^{\infty} \left(1-\lambda-s/2+\sqrt{2\lambda}s^{1/2}\right)_+^{\gamma} s^{b-1} ds \\
&= \frac{\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_0^{\infty} \left(1-\left(\sqrt{\lambda}-\sqrt{s}\right)^2\right)_+^{\gamma} s^{b-1} ds \\
&= \frac{2\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_0^{\infty} \left(1-\left(\sqrt{\lambda}-s\right)^2\right)_+^{\gamma} s^{2b-1}\, ds \\
&= \frac{2\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_{-\min(1,\sqrt{\lambda})}^{1} \left(1-s^2\right)^{\gamma} |s+\sqrt{\lambda}|^{2b-1}\, ds.
\end{aligned}
$$

Similarly, if $\lambda_1 = -\sqrt{2\lambda}$ and $\lambda_2 = \lambda \geq 0$ we obtain

$$\begin{aligned} C\Big(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \Big(-\sqrt{2\lambda}, \lambda\Big)\Big) &= \frac{\Gamma(b+1+\gamma)}{2^b\,\Gamma(\gamma+1)\Gamma(b)} \int_0^\infty \Big(1-\lambda-s/2-\sqrt{2\lambda}s^{1/2}\Big)_+^\gamma s^{b-1} ds \\ &= \frac{2\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_{-\infty}^0 \Big(1-\Big(\sqrt{\lambda}-s\Big)^2\Big)_+^\gamma |s|^{2b-1} ds \\ &= \frac{2\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_{-1}^{-\min(1,\sqrt{\lambda})} \big(1-s^2\big)^\gamma\, |s+\sqrt{\lambda}|^{2b-1} ds. \end{aligned}$$

Consequently

$$\begin{aligned} C^* &:= C\left(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \left(\sqrt{2\lambda}, \lambda\right)\right) + C\left(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \left(-\sqrt{2\lambda}, \lambda\right)\right) \\ &= \frac{2\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_{-1}^1 \big(1-s^2\big)\, |s+\sqrt{\lambda}|^{2b-1}\, ds. \end{aligned}$$

For $b = 1/2$ we simply get $C^* = 2$. Note in passing that $\lambda = 0$ implies $C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, (0,0)) = 1$.

Example 6 Consider $F, \mathcal{I}_1, \mathcal{I}_2, q_1, q_2, \boldsymbol{X}, \boldsymbol{Y}$ as in Example 2 and let $a_{n1}, a_{n2}, c_n \in (0,\infty), n \geq 1$ be constants such that

$$\lim_{n\to\infty} a_{n1} = \lim_{n\to\infty} (1-a_{n2}) = \lim_{n\to\infty} c_n = 0.$$

Suppose further that F has upper endpoint $x_F = 1$. Then we have

$$\begin{aligned} &\frac{\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > 1-c_n\}}{\boldsymbol{P}\{X_2 > 1-c_n\}} \\ &\quad= \boldsymbol{P}\{\mathcal{I}_1 = 1\}\,\boldsymbol{P}\{\mathcal{I}_2 = 1\}\, \frac{\boldsymbol{P}\{a_{n1}X_1 + a_{n2}X_2 > 1-c_n\}}{\boldsymbol{P}\{X_2 > 1-c_n\}} \\ &\qquad+ \boldsymbol{P}\{\mathcal{I}_1 = -1\}\,\boldsymbol{P}\{\mathcal{I}_2 = 1\}\, \frac{\boldsymbol{P}\{-a_{n1}X_1 + a_{n2}X_2 > 1-c_n\}}{\boldsymbol{P}\{X_2 > 1-c_n\}} + o(1),\ n \to \infty. \end{aligned}$$

If further $F, a_{n1}, a_{n2}, c_n, n \geq 1$ satisfy the assumptions of Theorem 4.1 with

$$p_1 = r_1 = 2, \quad \lambda_1 = \sqrt{2\lambda}, \quad \lambda_2 = \lambda \geq 0,$$

then in view of Example 5

$$\begin{aligned} &\frac{\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > 1-c_n\}}{\boldsymbol{P}\{X_2 > 1-c_n\}} \\ &\quad= q_2 \frac{2\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \left[q_1 \int_0^\infty \left(1-\left(\sqrt{\lambda}-s\right)^2\right)_+^\gamma s^{2b-1}\, ds \right. \\ &\qquad \left. +(1-q_1) \int_{-\infty}^0 \left(1-\left(\sqrt{\lambda}-s\right)^2\right)_+^\gamma |s|^{2b-1}\, ds \right] + o(1), \quad n \to \infty. \end{aligned}$$

Setting $q_1 := 1/2$ and assuming $q_2 \in (0, 1]$ we may write

$$\frac{\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > 1 - c_n\}}{\boldsymbol{P}\{X_2 > 1 - c_n\}}$$
$$= q_2 \frac{\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_{-1}^{1} \left(1 - s^2\right)^{\gamma} |s + \sqrt{\lambda}|^{2b-1}\, ds + o(1), \quad n \to \infty.$$

Consequently, if further $b = 1/2$, then we have

$$\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > 1 - c_n\} = (1 + o(1))\boldsymbol{P}\{Y_2 > 1 - c_n\}, \quad n \to \infty. \tag{5.6}$$

If (X_1, X_2) is a bivariate spherical random vector, then (5.6) follows immediately by (5.2).

Next, applying Theorem 6.2 in Hashorva (2007c) (recall (5.4)) we have

$$\boldsymbol{P}\{X_2 > 1 - c_n\} = (1 + o(1)) \frac{\Gamma(\gamma+1)\Gamma(a+b)}{\Gamma(a)\Gamma(\gamma+b+1)} (2c_n)^b [1 - F(1 - c_n)], \ n \to \infty,$$

hence we obtain

$$\begin{aligned}
&\boldsymbol{P}\{a_{n1}Y_1 + a_{n2}Y_2 > 1 - c_n\} \\
&\quad = (1 + o(1)) q_2 \frac{\Gamma(b+1+\gamma)}{\Gamma(\gamma+1)\Gamma(b)} \int_{-1}^{1} (1 - s^2)^{\gamma} |s + \sqrt{\lambda}|^{2b-1}\, ds \\
&\qquad \times \frac{\Gamma(\gamma+1)\Gamma(a+b)}{\Gamma(a)\Gamma(\gamma+b+1)} (2c_n)^b [1 - F(1 - c_n)] + o(1) \\
&\quad = (1 + o(1)) c_n^b [1 - F(1 - c_n)] q_2 2^b \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \\
&\qquad \times \int_{-1}^{1} (1 - s^2)^{\gamma} |s + \sqrt{\lambda}|^{2b-1}\, ds + o(1), \quad n \to \infty,
\end{aligned}$$

which is proved in Theorem 2.1 in Hashorva (2007b) for $q_2 = 1/2$ (set ρ_i to $\sqrt{\lambda}$ in Eq. 2.5 therein).

Example 7 Let $F, \boldsymbol{X}, \boldsymbol{Y}$ be as in Example 2, and let $\left(V_{n1}^{(j)}, V_{n2}^{(j)}\right)$, $M_{nl}, a_{n,kl}, k$, $l = 1, 2, 1 \le j \le n, n \ge 1$ be as in Example 4. Assume that F is in the max-domain of attraction of $\Psi_{\gamma}, \gamma \in (0, \infty)$ and $x_F = 1$. Suppose that the condition (3.12) holds with

$$\lambda_{11} = \lambda_{12} = 0, \quad \lambda_{21} = \sqrt{2\tau}, \quad \lambda_{22} = \tau \in [0, \infty)$$

and set

$$p_1 = r_1 = 2, \quad q_1 = q_2 = b = 1/2, \quad c_n := 1 - G^{-1}(1 - 1/n), \quad n > 1,$$

where G^{-1} is the quantile function of Y_1. Example 5 and (5.6) imply

$$\left(\frac{M_{n1} - 1}{c_n}, \frac{M_{n2} - 1}{c_n}\right) \xrightarrow{d} (\mathcal{M}_1, \mathcal{M}_2) \sim H, \quad n \to \infty,$$

where $\mathcal{M}_i \sim \Psi_{\gamma+1/2}, i = 1, 2$ and the distribution function H is defined by

$$-\ln(H(x, y)) = |x|^{\gamma+1/2} + |y|^{\gamma+1/2} - A(x, y), \quad x < 0, y < 0,$$

with

$$A(x, y) = \frac{\Gamma(1/2 + 1 + \gamma)}{2\sqrt{2}\Gamma(\gamma + 1)\Gamma(1/2)} \times \int_0^\infty \Big[\min\Big((|x| - s/2)_+, \Big(|y| - \tau - s/2 + \sqrt{2\tau s}\Big)_+\Big)^\gamma + \min\Big((|x| - s/2)_+, \Big(|y| - \tau - s/2 - \sqrt{2\tau s}\Big)_+\Big)^\gamma\Big] s^{-1/2}\, ds.$$

The case $\boldsymbol{Y}$ is elliptically distributed is treated initially in Gale (1980), Eddy and Gale (1981) assuming that F is absolutely continuous with algebraic tails. Hashorva (2006b, 2008) considers some general F in the Weibull max-domain of attraction.

6 Related asymptotics and proofs

Lemma 6.1 *Let F be a distribution function on $\mathbb{R}$ with upper endpoint $x_F \in (0, \infty]$, and $g_u(\boldsymbol{x}), u \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^k, k \geq 2$ be measurable functions such that*

$$\lim_{u \uparrow x_F} g_u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in [0, \infty)^k \tag{6.1}$$

and for all u in a left neighbourhood of x_F

$$g_u(\boldsymbol{x}) \geq \sum_{i=1}^k \tau_i x_i^{b_i}, \quad \forall \boldsymbol{x} \in [0, \infty)^k, \tag{6.2}$$

with $b_i, \tau_i, 1 \leq i \leq k$ some positive constants. If $F \in MDA(\Lambda, w)$, then we have for given constants $c_i \geq 0$, $p_i \in (-1, \infty)$, $1 \leq i \leq k$

$$\int_{c_1}^{\delta_u} \cdots \int_{c_k}^{\delta_u} \prod_{i=1}^k x_i^{p_i} \Big[1 - F(u + g_u(\boldsymbol{x})/w(u))\Big] dx_1 dx_2 \cdots dx_k = (1 + o(1)) K[1 - F(u)], \quad u \uparrow x_F, \tag{6.3}$$

where $\delta_u := w(u)(x_F - u)$, $u > 0$ with $\delta_u := \infty$ if $x_F = \infty$ and K is defined by

$$K := \int_{c_1}^\infty \cdots \int_{c_k}^\infty \prod_{i=1}^k x_i^{p_i} \exp(-g(\boldsymbol{x}))\, dx_1 dx_2 \cdots dx_k \in (0, \infty).$$

Proof of Lemma 6.1 First note that

$$\int_{c_1}^{\delta_u} \cdots \int_{c_k}^{\delta_u} \prod_{i=1}^{k} x_i^{p_i} \exp(-g(\boldsymbol{x}))\, dx_1 dx_2 \cdots dx_k$$
$$\leq \prod_{i=1}^{k} \left[\int_{c_i}^{\infty} s^{p_i} \exp\left(-\tau_i s^{b_i}\right)\, ds \right] =: K \in (0, \infty).$$

By the assumption on F (3.1) holds uniformly for x in compact sets of $I\!R$, consequently Consequently, Fatou's Lemma implies

$$\liminf_{u \uparrow x_F} \frac{1}{1 - F(u)} \int_{c_1}^{\delta_u} \cdots \int_{c_k}^{\delta_u} \prod_{i=1}^{k} x_i^{p_i} [1 - F(u + g_u(\boldsymbol{x})/w(u))]$$
$$\times\, dx_1 dx_2 \cdots dx_k \geq K.$$

The proof for the lim sup is not trivial. Suppose for simplicity that $b_i = 1 - c_i = \tau_i = 1, 1 \leq i \leq k$. When $x_F = \infty, k = 1, p_1 \geq 0$ the proof follows from Lemma 4.2 in Hashorva (2006a) (see also Berman 1992). Iterating we get the proof for $k \geq 1$. The case $x_F < \infty$ follows with similar arguments, see the proof of Lemma 4.2 in Hashorva (2006a). We consider next the case $p_1 \in (-1, 0)$, $k = 1$. Since $F \in MDA(\Lambda, w)$ we have

$$\lim_{u \uparrow x_F} \frac{1 - F(u + x/w(u))}{\exp(-x)(1 - F(u))} = 1$$

holds uniformly for $x \in [0, 1]$. Hence, for any $\varepsilon > 0$, $x \in [0, 1]$ we have

$$\left| \frac{1 - F(u + x/w(u))}{1 - F(u)} - \exp(-x) \right| < \varepsilon \exp(-x)$$

implying

$$\int_0^1 x^{p_1} [1 - F(u + x/w(u))]\, dx = (1 + o(1))[1 - F(u)] \int_0^1 x^{p_1} \exp(-x)\, dx, \;\; u \uparrow x_F,$$

with $\int_0^1 x^{p_1} \exp(-x)\, dx \in (0, \infty)$. The result for $p_1 \geq 0$ and (3.1) yield further

$$\int_1^{\infty} x^{p_1} [1 - F(u + x/w(u))]\, dx = (1 + o(1))[1 - F(u)] \int_1^{\infty} x^{p_1} \exp(-x)\, dx, \;\; u \uparrow x_F,$$

hence

$$\int_0^{\infty} x^{p_1} [1 - F(u + x/w(u))]\, dx = (1 + o(1))[1 - F(u)] \Gamma(p_1 + 1), \quad u \uparrow x_F.$$

The proof for $k \geq 1$ follows by partial integration. □

The following corollary is immediate:

Corollary 6.2 *Let* $F, w, \delta_u, u > 0$ *be as in Lemma* 6.1, *then we have for* $p_i \in (-1, \infty), 1 \leq i \leq k$

$$\lim_{u \uparrow x_F} \frac{1}{1 - F(u)} \int_0^{\delta_u} \cdots \int_0^{\delta_u} \prod_{i=1}^{k} x_i^{p_i} \left[1 - F(u + \sum_{i=1}^{k} x_i / w(u)) \right] dx_1 dx_2 \cdots dx_k$$
$$= \prod_{i=1}^{k} \Gamma(1 + p_i) \in (0, \infty). \tag{6.4}$$

Proof of Theorem 3.1 The proof is similar to the proof of Theorem 12.3.1 in Berman (1992), therefore we omit some details. Define next

$$\chi(x_1, \ldots, x_{k-1}) := \sum_{i=1}^{k-1} a_{ni} (1 - x_i)^{1/r_i} \prod_{j=1}^{i-1} x_j + a_{nk} \prod_{j=1}^{k-1} x_j^{1/p_j},$$
$$(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}, \quad n \geq 1,$$
$$I_n(s, t) := \int_s^t \cdots \int_x^y \boldsymbol{P}\{R\chi(x_1, \ldots, x_{k-1}) > y_n\} \, dB_1(x_1) \cdots dB_{k-1}(x_{k-1}),$$
$$0 \leq s < t \leq 1, n \geq 1, \tag{6.5}$$

and $B_i(x), x \in [0, 1], i \leq k - 1$ the Beta distribution function with positive parameters a_i, b_i. By the assumptions

$$\sum_{i=1}^{k} a_{ni} X_i \stackrel{d}{=} R\chi(V_1, \ldots, V_{k-1}), \quad n \geq 1.$$

For all $x_i \in \mathbb{R}, i \leq k - 1$ (3.3) implies

$$\chi(x_1, \ldots, x_{k-1}) = (1 + o(1)) \prod_{j=1}^{k-1} x_j^{1/p_j}, \quad n \to \infty. \tag{6.6}$$

Case $x_F = \infty$: $F \in MDA(\Lambda, w)$ implies that (see e.g., Resnick 2008)

$$\lim_{u \to \infty} \frac{1 - F(xu)}{1 - F(u)} = 0, \quad \forall x > 1.$$

For $\varepsilon, \delta \in (0,1)$ with $\varepsilon < \delta$ and all large n utilising (6.6) we obtain

$$\begin{aligned}
&\frac{I_n(0,1-\delta)}{I_n(1-\varepsilon,1)}\\
&= \frac{\int_0^{1-\delta}\cdots\int_0^{1-\delta} \boldsymbol{P}\{R\chi(x_1,\ldots,x_{k-1}) > y_n\}\, dB_1(x_1)\cdots dB_{k-1}(x_{k-1})}{\int_{1-\varepsilon}^{1}\cdots\int_{1-\varepsilon}^{1} \boldsymbol{P}\{R\chi(x_1,\ldots,x_{k-1}) > y_n\}\, dB_1(x_1)\cdots dB_{k-1}(x_{k-1})}\\
&\le \frac{\boldsymbol{P} R > (1+o(1))y_n(1-\delta)^{-\sum_{i=1}^{k-1} 1/p_i}}{\boldsymbol{P}\left\{R > (1+o(1))y_n(1-\varepsilon)^{-\sum_{i=1}^{k-1} p_i}\right\}\int_{1-\varepsilon}^{1}\cdots\int_{1-\varepsilon}^{1} dB_1(x_1)\cdots dB_{k-1}(x_{k-1})} \to 0,\\
&\quad n\to\infty.
\end{aligned}$$

Hence for any $\varepsilon \in (0,1)$ we may write

$$I_n(0,1) = (1+o(1))I_n(1-\varepsilon,1), \quad n\to\infty.$$

Transforming the variables we obtain for all n large (set $h_n := y_n w(y_n), n \ge 1$)

$$\begin{aligned}
I_n(1-\varepsilon,1) &= (1+o(1))\int_{1-\varepsilon}^{1}\cdots\int_{1-\varepsilon}^{1} \boldsymbol{P}\left\{R\chi(x_1,\ldots,x_{k-1}) > y_n\right\} dB_1(x_1)\cdots dB_{k-1}(x_{k-1})\\
&= (1+o(1))\int_0^{\varepsilon h_n}\cdots\int_0^{\varepsilon h_n} \boldsymbol{P}\left\{R\chi(1-s_1/h_n,\ldots,1-s_{k-1}/h_n) > y_n\right\}\\
&\quad \times dB_1(1-s_1/h_n)\cdots dB_{k-1}(1-s_{k-1}/h_n)\\
&=: (1+o(1))\int_0^{\varepsilon h_n}\cdots\int_0^{\varepsilon h_n} \psi_n(s_1,\ldots,s_{k-1})\\
&\quad \times dB_1(1-s_1/h_n)\cdots dB_{k-1}(1-s_{k-1}/h_n).
\end{aligned}$$

By (2.6) and (3.11) we have

$$\begin{aligned}
&\psi_n(s_1,\ldots,s_{k-1})\\
&\quad = \boldsymbol{P}\left\{R > y_n + (1+o(1))\left[\lambda_k + \sum_{i=1}^{k-1} s_i/p_i - \sum_{i=1}^{k-1}\lambda_i s_i^{1/r_i}\right]/w(y_n)\right\}, \quad n\to\infty
\end{aligned}$$

and

$$\lim_{n\to\infty} h_n^{b_i}\Big[1 - B_i(1-s/h_n)\Big] = \frac{\Gamma(a_i+b_i)}{b_i\Gamma(a_i)\Gamma(b_i)} s^{b_i}, \quad \forall s \ge 0,\ 1 \le i \le k-1. \tag{6.7}$$

Furthermore we have the convergence of the density functions

$$\lim_{n\to\infty} h_n^{b_i-1} B_i'(1-s/h_n) = \frac{\Gamma(a_i+b_i)}{\Gamma(a_i)\Gamma(b_i)} s^{b_i-1}, \quad \forall s \ge 0, 1 \le i \le k-1, \tag{6.8}$$

with B_i' the density function of $B_i, i \le k$. Hence we may further write

$$\begin{aligned}&\int_0^{\varepsilon h_n} \cdots \int_0^{\varepsilon h_n} \psi_n(s_1,\ldots,s_{k-1})\, dB_1(1-s_1/h_n)\cdots dB_{k-1}(1-s_{k-1}/h_n)\\ &\quad= (1+o(1)) h_n^{\sum_{i=1}^{k-1} b_i} \int_0^{\varepsilon h_n} \cdots \int_0^{\varepsilon h_n} \psi_n(s_1,\ldots,s_{k-1})\\ &\qquad\times \prod_{i=1}^{k-1} s_i^{b_i-1} \frac{\Gamma(a_i+b_i)}{\Gamma(a_i)\Gamma(b_i)}\, ds_1\cdots ds_{k-1}, n\to\infty.\end{aligned}$$

Next, Theorem 6.2 in Hashorva (2007c) implies

$$\frac{\boldsymbol{P}\{X_k > y_n\}}{\boldsymbol{P}\{R > y_n\}} = (1+o(1)) \prod_{i=1}^{k-1} \frac{\Gamma(a_i+b_i)}{\Gamma(a_i) p_i^{-b_i}} h_n^{-\sum_{i=1}^{k-1} b_i}, \quad n\to\infty. \tag{6.9}$$

Since F is in the Gumbel max-domain of attraction

$$\lim_{n\to\infty} \frac{\psi_n(s_1,\ldots,s_{k-1})}{\boldsymbol{P}\{R > y_n\}} = \exp\left(-\lambda_k - \sum_{i=1}^{k-1}\left[s_i/p_i - \lambda_i s_i^{1/r_i}\right]\right), \quad \forall s_i \in I\!R, \quad i \le k-1$$

holds uniformly for $s_i, i \le k-1$ in compact sets of $I\!R$. By the assumptions $\lambda_i = 0$ if $r_i \le 1$ for some $i \le k$. Hence for all $s_i \in [0, \varepsilon h_n], i \le k-1$ we have that $g_n(s_1,\ldots,s_{k-1}) := h_n[1/\chi(1-s_1/h_n,\ldots,1-s_{k-1}/h_n)-1]$ satisfies the assumptions of Lemma 6.1 for all large n (setting $u := y_n$), consequently applying Lemma 6.1 we obtain

$$\begin{aligned}&\frac{\boldsymbol{P}\left\{\sum_{i=1}^k a_{ni} X_i > y_n\right\}}{\boldsymbol{P}\{X_k > y_n\}}\\ &\quad= (1+o(1)) \frac{I_n(1-\varepsilon,1)}{\boldsymbol{P}\{R > y_n\}} \frac{\boldsymbol{P}\{R > y_n\}}{\boldsymbol{P}\{X_k > y_n\}}\\ &\quad= (1+o(1)) \prod_{i=1}^{k-1} \frac{\Gamma(a_i) p_i^{-b_i}}{\Gamma(a_i+b_i)} \prod_{j=1}^{k-1}\left[\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)\Gamma(b_j)}\right]\\ &\qquad\times \int_0^\infty \cdots \int_0^\infty \exp\left(\sum_{i=1}^{k-1}\left[\lambda_i s_i^{1/r_i} - s_i/p_i\right] - \lambda_k\right) \prod_{j=1}^{k-1} s_j^{b_j-1}\, ds_1\cdots ds_{k-1}\end{aligned}$$

$$= (1+o(1)) \prod_{j=1}^{k-1} \left[\frac{1}{p_j^{b_j} \Gamma(b_j)} \right] \int_0^\infty \cdots \int_0^\infty \exp\left(\sum_{i=1}^{k-1} \left[\lambda_i s_i^{1/r_i} - s_i/p_i \right] - \lambda_k \right)$$

$$\times \prod_{j=1}^{k-1} s_j^{b_j-1}\, ds_1 \cdots ds_{k-1}$$

$$= (1+o(1)) \exp(-\lambda_k) \prod_{j=1}^{k-1} \left[\frac{1}{p_j^{b_j} \Gamma(b_j)} \int_0^\infty \exp\left(\lambda_j s^{1/r_j} - s/p_j \right) s^{b_j-1}\, ds \right],$$

$$n \to \infty.$$

Case $x_F < \infty$: The proof for this case follows with similar arguments using further (3.10). □

Proof of Corollary 3.2 The Gumbel max-domain of attraction assumption on F implies that the scaling function satisfies the self-neglecting property (see e.g., Resnick 2008), i.e.,

$$\lim_{u \uparrow x_F} \frac{w(u + x/w(u))}{w(u)} = 1 \tag{6.10}$$

holds uniformly for x in compact sets of $\mathbb{R}$. Consequently utilising (3.7) it follows that X_k has distribution function G in the max-domain of attraction of Λ with the same scaling function w. Define next $y_n := G^{-1}(1-1/n), n > 1$, with G^{-1} the inverse of G. Applying Theorem 3.1 we obtain for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\boldsymbol{P}\{Z_{n1} > y_n + x/w(y_n)\}}{\boldsymbol{P}\{X_k > y_n\}} = \exp(-x) \lim_{n\to\infty} \frac{\boldsymbol{P}\{Z_{n1} > y_n + x/w(y_n)\}}{\boldsymbol{P}\{X_k > y_n + x/w(y_n)\}}$$

$$= \exp(-x) K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}).$$

Consequently, setting $d_n := y_n, c_n := 1/w(d_n), n > 1$ for any $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} n\boldsymbol{P}\left\{ Z_{n1} > c_n(x + \ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}))) + d_n \right\}$$

$$= \exp(-(\ln(K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda})) + x)) K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) = \exp(-x),$$

hence the proof is complete.

Proof of Theorem 3.4 Define for $n \in \mathbb{N}$, $1 \le l \le k$ the constants y_{nl} by

$$y_{nl} := c_n(x_l + K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l)) + d_n, \quad x_l \in \mathbb{R},$$

with $c_n, d_n, n > 1$ as in Corollary 3.2. By (3.10) and (6.10) as $n \to \infty$

$$y_{nl} \uparrow x_F, \quad \text{and } y_{nl} w(y_{nl}) = (1+o(1)) d_n w(d_n) = (1+o(1)) d_n/c_n, \quad 1 \le l \le k.$$

Hence in view of the assumption (3.12) applying Corollary 3.2 for $1 \le l \le k$ we obtain

$$\frac{\max_{1\le j\le n} X_{nl}^{(j)} - d_n}{c_n} - K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l) \xrightarrow{d} \mathcal{M}_l \sim \Lambda, \quad n \to \infty,$$

In order to complete the proof we need to determine (see Lemma 4.1.3 of Falk et al. 2004)

$$\lim_{n\to\infty} n\boldsymbol{P}\left\{X_{nl}^{(1)} > c_n(x_l + K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l)) + d_n, \forall l \in L\right\}$$

for any $L \subset \{1, \ldots, k\}$ with at least 2 elements. Proceeding as in the proof of Theorem 3.1 we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{X_{nl}^{(1)} > c_n(x_l + K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l)) + d_n, \forall l \in L\right\}\\
&\quad = \lim_{n\to\infty} \frac{1}{\boldsymbol{P}\{X_k > d_n\}} \boldsymbol{P}\left\{\sum_{j=1}^{k} a_{n,lj} X_j > c_n(x_l + K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l)) + d_n, \forall l \in L\right\}\\
&\quad = \left[\prod_{j=1}^{k-1} \frac{1}{p_j^{b_j}\Gamma(b_j)}\right] \int_0^\infty \cdots \int_0^\infty \exp\left(\min_{l\in L}\left[\sum_{j=1}^{k-1}\left[\lambda_{lj} s_j^{1/r_j} - s_j/p_j\right]\right.\right.\\
&\qquad \left.\left. - x_l - \lambda_{lk} - K(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}_l)\right]\right) \prod_{j=1}^{k-1} s_j^{b_j-1}\, ds_1 \cdots ds_{k-1}.
\end{aligned}$$

Hence the proof follows easily. □

Proof of Theorem 4.1 Using the same notation as in the proof of Theorem 3.1 we may write for all large n by transforming the variables

$$\begin{aligned}
I_n(0,1) &= \int_0^1 \cdots \int_0^1 \boldsymbol{P}\{R\chi(x_1, \ldots, x_{k-1}) > 1 - 1/h_n\}\, dB_1(x_1) \cdots dB_{k-1}(x_{k-1})\\
&= \int_0^{h_n} \cdots \int_0^{h_n} \boldsymbol{P}\{R\chi(1 - s_1/h_n, \ldots, 1 - s_{k-1}/h_n) > 1 - 1/h_n\}\\
&\quad \times dB_1(1 - s_1/h_n), \ldots, dB_{k-1}(1 - s_{k-1}/h_n)\\
&=: \int_0^{h_n} \cdots \int_0^{h_n} \psi_n(s_1, \ldots, s_{k-1})\, dB_1(1 - s_1/h_n), \ldots, dB_{k-1}(1 - s_{k-1}/h_n).
\end{aligned}$$

Since $\lim_{n\to\infty} h_n = \infty$ the assumptions on $a_{ni}, i \le k, n \ge 1$ imply

$$\begin{aligned}
&\psi_n(s_1, \ldots, s_{k-1})\\
&\quad = (1 + o(1))\boldsymbol{P}\left\{R > 1 - \left[1 - (1 + o(1))\left(\lambda_k + \sum_{i=1}^{k-1} s_i/p_i - \sum_{i=1}^{k-1} \lambda_i s_i^{1/r_i}\right)\right]/h_n\right\},\\
&\qquad n \to \infty
\end{aligned}$$

uniformly for $s_i, i \le k-1$ in compact sets of $(0, \infty)$. Furthermore, (4.1) yields $\forall s_i \in (0, \infty), 1 \le i \le k-1$

$$\lim_{n\to\infty} \frac{\psi_n(s_1, \ldots, s_{k-1})}{\boldsymbol{P}\{R > 1 - 1/h_n\}} = \left(1 - \lambda_k - \sum_{i=1}^{k-1} s_i/p_i + \sum_{i=1}^{k-1} \lambda_i s_i^{1/r_i}\right)_+^{\gamma},$$

where $(x)_+ := \max(x, 0)$, $x \in I\!R$. Now, by (6.8) and the fact that the integrand equals 0 for all $s_i, i \le k-1$ large, we obtain applying Lemma 4.2 in Hashorva (2007a)

$$\lim_{n\to\infty} \frac{h_n^{\sum_{i=1}^{k-1} b_i} I_n(0,1)}{\boldsymbol{P}\{R > 1-1/h_n\}}$$
$$= \left[\prod_{i=1}^{k-1} \frac{\Gamma(a_i+b_i)}{\Gamma(a_i)\Gamma(b_i)}\right] \int_0^\infty \cdots \int_0^\infty \left(1-\lambda_k - \sum_{i=1}^{k-1} s_i/p_i + \sum_{i=1}^{k-1} \lambda_i s_i^{1/r_i}\right)_+^\gamma$$
$$\times \prod_{i=1}^{k-1} s_i^{b_i+\gamma-1}\, ds_1 \cdots ds_{k-1}.$$

Theorem 6.2 in Hashorva (2007c) implies

$$\lim_{n\to\infty} \frac{\boldsymbol{P}\{X_k > 1-1/h_n\}}{\boldsymbol{P}\{R > 1-1/h_n\}} = h_n^{-\sum_{i=1}^{k-1} b_i} \prod_{i=1}^{k-1} p_i^{b_i} \frac{\Gamma(a_i+b_i)\Gamma(\gamma_i^*-b_i)}{\Gamma(a_i)\Gamma(\gamma_i^*)},$$

where $\gamma_i^* := 1+\gamma+\sum_{j=i}^{k-1} b_j$. Consequently, as $n\to\infty$ we obtain

$$I_n(0,1) = (1+o(1))\boldsymbol{P}\{X_k > 1-1/h_n\}\left[\prod_{i=1}^{k-1} \frac{\Gamma(\gamma_i^*)}{p_i^{b_i}\Gamma(\gamma_i^*-b_i)\Gamma(b_i)}\right]$$
$$\times \int_0^\infty \cdots \int_0^\infty \left(1-\lambda_k - \sum_{i=1}^{k-1} s_i/p_i + \sum_{i=1}^{k-1} \lambda_i s_i^{1/r_i}\right)_+^\gamma \prod_{i=1}^{k-1} s_i^{b_i-1}\, ds_1 \cdots ds_{k-1}.$$

This completes the proof. □

Proof of Corollary 4.2 Since F is in the Weibull max-domain of attraction of Ψ_γ it follows by Theorem 4.1 that X_k has distribution function G in the max-domain of attraction of Ψ_δ, with $\delta := \gamma + \sum_{i=1}^{k-1} b_i > 0$. Let G^{-1} be the inverse of G and define $c_n := 1 - G^{-1}(1-1/n), n > 1$. Since $\lim_{n\to\infty} 1/c_n = \infty$, for any $x < 0$ utilising further Theorem 4.1 we obtain

$$\lim_{n\to\infty} n\boldsymbol{P}\left\{c_n^{-1}(Z_{n1}-1) > x\right\} = \lim_{n\to\infty} \frac{\boldsymbol{P}\left\{\sum_{j=1}^k a_{nj} X_j > 1 - c_n|x|\right\}}{\boldsymbol{P}\{X_k > 1-c_n\}}$$
$$= C(\gamma, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda x}^*)|x|^\delta,$$

with $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_k)^\top$ and $\boldsymbol{x}^* := (|x|^{1/r_1-1}, \ldots, |x|^{1/r_{k-1}-1}, |x|^{-1})$, hence the proof follows. □

Proof of Theorem 4.3 Applying Corollary 3.2 we obtain for $1 \le l \le k$

$$\frac{\max_{1\le j\le n} X_{nl}^{(j)} - 1}{c_n} \xrightarrow{d} \mathcal{M}_l \sim Q_{\gamma,\boldsymbol{b},\boldsymbol{p},\boldsymbol{r},\lambda_l}, \quad n\to\infty,$$

with $Q_{\gamma,\boldsymbol{b},\boldsymbol{p},\boldsymbol{r},\boldsymbol{\lambda}_l}$, $\boldsymbol{\lambda}_l := (\lambda_{l1}, \ldots, \lambda_{lk})^\top$ as in (4.6). By Lemma 4.1.3 in Falk et al. (2004) the proof follows if we determine the limit function of $n\boldsymbol{P}\left\{X_{nl}^{(1)} > 1 + c_n x_l, \forall l \in L\right\}$ for any $L \subset \{1, \ldots, k\}$ with at least 2 elements and $x_l < 0$, $1 \le l \le k$. Along the lines of the proof of Theorem 4.1 we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{X_{nl}^{(1)} > 1 - c_n|x_l|, \forall l \in L\right\}\\
&= \lim_{n\to\infty} \frac{1}{\boldsymbol{P}\{X_k > 1 - c_n\}} \boldsymbol{P}\left\{\sum_{j=1}^{k} a_{n,lj} X_j > 1 - c_n|x_l|, \forall l \in L\right\}\\
&= (1 + o(1)) \left[\prod_{i=1}^{k-1} \frac{\Gamma(\gamma_i^*)}{p_i^{b_i}\Gamma(\gamma_i^* - b_i)\Gamma(b_i)}\right]\\
&\times \int_0^\infty \cdots \int_0^\infty \min_{l\in L}\left(|x_l| - \lambda_{lk} - \sum_{i=1}^{k-1} s_i/p_i + \sum_{i=1}^{k-1} \lambda_{li} s_i^{1/r_i}\right)_+^{\gamma}\\
&\times \prod_{i=1}^{k-1} s_i^{b_i - 1}\, ds_1 \cdots ds_{k-1},
\end{aligned}
$$

where $\gamma_i^* := 1 + \gamma + \sum_{j=i}^{k-1} b_j$. Hence the result follows. □

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